Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Sound and Vibration 286 (2005) 69–79

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

A statistical nonlinearization technique for multi-degrees of freedom nonlinear systems under white noise excitations

Cho W.S. To

Department of Mechanical Engineering, N104 Walter Scott Engineering Center, University of Nebraska, Lincoln, NE 68588-0656, USA

Received 17 May 2004; received in revised form 22 September 2004; accepted 28 September 2004
Available online 12 January 2005

Abstract

A statistical nonlinearization (SNL) technique is proposed for the solution of the joint probability density function of general multi-degrees of freedom (mdof) nonlinear systems under stationary white noise excitations. The nonlinearities associated with damping and restoring forces as well as the intensities of excitations are not necessarily small. It is based on the application of the exact solution of the joint probability density function of a mdof nonlinear systems under stationary white noise excitations. This exact solution is different from that of Caughey in that in the present exact solution the ratios of damping coefficients to applied white noise excitations are not identical. The exact solution is also different from those of Cai and Lin, and Zhu and Huang in that the present exact solution is obtained directly from the theory of differential equations while that of Cai and Lin requires satisfaction of a relatively restrictive criterion. Furthermore, the Hamiltonian formulation is applied by Zhu and Huang. Their solution depends on the number of independent integrals of motion, for example. Results by applying the proposed technique for a general two-degrees of freedom nonlinear system are compared with those obtained by Monte Carlo simulation (MCS). It is concluded that the technique is accurate, simple to implement and is applicable to mdof systems with both nonlinear damping and nonlinear restoring forces.

E-mail address: cto2@unl.edu (C.W.S. To).

doi:10.1016/j.jsv.2004.10.006

1. Introduction

Various statistical nonlinearization (SNL) techniques exist in the literature for single-degree of freedom (sdof) nonlinear systems under stationary white noise excitation [1]. For two-degrees of freedom (tdof) nonlinear systems the author has presented two SNL techniques [2,3] for response analysis. In Ref. [2], the SNL technique for the tdof nonlinear system is based on the exact solution of Caughey [4,5] in which the ratios of coefficients of linear damping forces to intensities of stationary white noise excitations are identical. Therefore, the class of tdof nonlinear systems that can be solved by the SNL technique in Ref. [2] is not large. The SNL technique for tdof nonlinear system in Ref. [3] improved that in Ref. [2] so that the ratios of coefficients of linear damping forces to the intensities of stationary white noise excitations are not identical. Owing to its usefulness and simplicity as well as the fact that it includes a much wider class of tdof nonlinear systems, it is further developed and generalized for application to multi-degrees of freedom (mdof) nonlinear systems under stationary white noise excitations. As there is no exact solution for the joint probability density function of mdof systems with both nonlinear damping forces and nonlinear restoring forces, the presently proposed SNL technique can be applied to obtain the approximate solution.

The organization of the remaining part of this paper is as follows. Section 2 is concerned with the theoretical development and exact solution of the reduced Fokker–Planck–Kolmogorov (FPK) equation for mdof nonlinear systems in which the dampling forces are linear. However, the ratios of damping coefficients to intensities of white noise excitations are not equal and therefore this exact solution is not similar to those available in the literature. Section 3 deals with the development of the new SNL technique for a relatively wide class of mdof nonlinear systems in which the damping forces and restoring forces are nonlinear. Application is made in Section 4 of the SNL technique for the solution of a tdof system that has nonlinear damping and nonlinear stiffness terms. Computed results by applying the proposed SNL technique are compared with those from Monte Carlo simulation (MCS). The final section, Section 5, includes concluding remarks.

2. Exact solution of mdof nonlinear systems

Many solutions have been reported in the literature in regard to the exact joint stationary probability density functions of mdof nonlinear systems under stationary random excitations. These solutions hinge around a generalized stationary potential that is proportional to the total energy of the system and its kinetic energies among different modes are identically distributed. The latter is known as equipartitioning of energy in the field of statistical mechanics. Typical results can be found, for example, in the publications of Caughey [4,5], Lin and Cai [6], Soize [7], Zhu and Lin [8], and To [2]. It may be appropriate to note that more recently there are other exact solutions to the reduced FPK equation for mdof nonlinear systems in the literature [9,10]. Strictly speaking, dampings considered in these two references are linear since their coefficients are functions of total energies which are constant in the time domain, and solutions are hinged on various conditions. For example, in Ref. [10], a Hamiltonian formulation is adopted for the solution of non-resonant and resonant cases. The existence of action and angle variables of the integrable part of the Hamiltonian system is assumed. Furthermore, in the resonant case a restriction on the diffusion coefficients of the system is imposed. Without such a restriction on the diffusion coefficients solution cannot be found.

In this section, an improved solution of the joint stationary probability density function of an mdof nonlinear system under stationary white noise excitations is presented. The solution is free from the limitation of Refs. [4,5] that the ratios of coefficients of linear damping forces to intensities of white noise excitations are being equal. It is direct and simple compared with those in Refs. [9,10].

Consider the mdof nonlinear system governed by the equations of motion

$$\ddot{y}_i + \alpha_{ii}\dot{y}_i + g_i(y_1, y_2, \ldots, y_n) = w_i(t), \tag{1}$$

where $w_i(t),\ i = 1, 2, \ldots, n$ are the zero mean Gaussian white noise excitations with

$$\langle w_i(t_1)w_i(t_2)\rangle = 2\pi S_i\delta(t_1 - t_2) = 2D_i\delta(t_1 - t_2),$$
$$\langle w_i(t_1)w_j(t_2)\rangle = 0, \quad i \neq j,$$

in which S_i are the spectral densities of the Gaussian white noises and the angular brackets denote the ensemble average or mathematical expectation.

Writing $x_1 = y_1,\ x_2 = y_2, \ldots,\ x_n = y_n,\ x_{n+1} = \dot{y}_1,\ x_{n+2} = \dot{y}_2, \ldots,\ x_{2n} = \dot{y}_n$, then Eq. (1) can be written in the state space form as

$$\begin{aligned}
\dot{x}_1 &= x_{n+1},\\
\dot{x}_2 &= x_{n+2},\\
&\vdots\\
\dot{x}_n &= x_{2n},\\
\dot{x}_{n+1} &= -\alpha_{11}x_{n+1} - g_1(x_1, x_2, \ldots, x_n) + w_1(t),\\
\dot{x}_{n+2} &= -\alpha_{22}x_{n+2} - g_2(x_1, x_2, \ldots, x_n) + w_2(t),\\
&\vdots\\
\dot{x}_{2n} &= -\alpha_{nn}x_{2n} - g_n(x_1, x_2, \ldots, x_n) + w_n(t).
\end{aligned} \tag{2}$$

The stationary FPK equation for the mdof nonlinear system becomes

$$\sum_{i=1}^{n}\left(D_i\,\frac{\partial^2 p}{\partial x_{n+i}^2} - \frac{\partial(x_{n+i}p)}{\partial x_i} + \frac{\partial}{\partial x_{n+i}}\,\{[\alpha_{ii}x_{n+i} + g_i(x_1, x_2, \ldots, x_n)]p\}\right) = 0. \tag{3}$$

Eq. (3) can be re-arranged as

$$\sum_{i=1}^{n}\left[\frac{\partial}{\partial x_{n+i}}\left(D_i\,\frac{\partial p}{\partial x_{n+i}} + \alpha_{ii}x_{n+i}p\right)\right] = \sum_{i=1}^{n}\left(x_{n+i}\,\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_{n+i}}\,[g_i(x_1, x_2, \ldots, x_n)p]\right). \tag{4}$$

The joint stationary probability density function (jspdf) $p(x_1, x_2, \ldots, x_{2n})$ or simply p is a solution of Eq. (4) if p satisfies the following equations:

$$D_i\,\frac{\partial p}{\partial x_{n+i}} + \alpha_{ii}x_{n+i}p = 0, \quad i = 1, 2, \ldots, n \tag{5}$$

and

$$\sum_{i=1}^{n}\left(x_{n+i}\frac{\partial p}{\partial x_i}-\frac{\partial}{\partial x_{n+i}}\left[g_i(x_1,x_2,\ldots,x_n)p\right]\right)=0. \tag{6}$$

By virtue of Eq. (5), the general solution of the jspdf $p(x_1,x_2,\ldots,x_{2n})$ can be shown to be

$$p=q(x_1,\ldots,x_n)\mathrm{e}^{-1/2\sum_{i=1}^{n}\beta_i x_{n+i}^2}, \tag{7}$$

where $q(x_1,\ldots,x_n)$ or simply q is a function of $x_1,x_2,\ldots,$ and x_n, and $\beta_i=\alpha_{ii}/D_i$.

Substituting Eq. (5) into Eq. (6), one has

$$\sum_{i=1}^{n}\left(x_{n+i}\frac{\partial p}{\partial x_i}+\beta_i x_{n+i}g_i(x_1,x_2,\ldots,x_n)p\right)=0. \tag{8}$$

By applying Eq. (7), Eq. (8) becomes

$$\sum_{i=1}^{n}\left(x_{n+i}\frac{\partial q}{\partial x_i}+\beta_i x_{n+i}g_i(x_1,x_2,\ldots,x_n)q\right)=0. \tag{9}$$

Since x_{n+i} are linearly independent, Eq. (9) reduces to

$$\sum_{i=1}^{n}\left(\frac{\partial q}{\partial x_i}+\beta_i g_i(x_1,x_2,\ldots,x_n)q\right)=0. \tag{10}$$

By virtue of Eq. (10), one has

$$q=q_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\mathrm{e}^{-\int\beta_i g_i\,\mathrm{d}x_i}, \tag{11}$$

with $i=1$ or 2 ... or n.

After some algebraic manipulation, one can show that there exists a function $U(x_1,x_2,\ldots,x_n)$ such that

$$\mathrm{d}U=\beta_1 g_1\,\mathrm{d}x_1+\beta_2 g_2\,\mathrm{d}x_2+\cdots+\beta_n g_n\,\mathrm{d}x_n, \tag{12a}$$

which can be expressed as

$$\mathrm{d}U=\frac{\partial U}{\partial x_1}\mathrm{d}x_1+\frac{\partial U}{\partial x_2}\mathrm{d}x_2+\cdots+\frac{\partial U}{\partial x_n}\mathrm{d}x_n, \tag{12b}$$

where

$$\frac{\partial U}{\partial x_i}=\beta_i g_i,\quad i=1,2,\ldots,n. \tag{12c}$$

By making use of Eq. (12), one can show that the necessary and sufficient condition for Eq. (12a) to be an exact differential equation is

$$\beta_1\frac{\partial^{n-1}g_1}{\partial x_2\partial x_3\ldots\partial x_n}=\beta_2\frac{\partial^{n-1}g_2}{\partial x_1\partial x_3\ldots\partial x_n}=\ldots=\beta_n\frac{\partial^{n-1}g_n}{\partial x_1\partial x_2\ldots\partial x_{n-1}}. \tag{13}$$

By virtue of Eqs. (7), (10), (11) and (13), one can obtain

$$p=C\mathrm{e}^{-\phi}, \tag{14}$$

in which C is the normalization constant and

$$\phi = \frac{1}{2}\left(\sum_{i=1}^{n} \beta_i x_{n+i}^2\right) + \int \beta_i g_i \, \mathrm{d}x_i,$$

where i in the second term on the right-hand side (rhs) can take the value of 1 or $2 \ldots$ or n.

Remark 2.1. Eq. (14) is the basis of the presently proposed SNL technique. The stationary potential in Eq. (14) is different from those presented by Caughey [4,5], Lin and Cai [6], Soize [7], Zhu and Lin [8], To [2], Cai and Lin [9], and Zhu and Huang [10]. The major difference between the present solution and those in Refs. [2,4–8] is the fact that in the present solution β_i are, in general, not equal. The present solution is also different from those of Refs. [9,10] in that in the present solution application of the theory of differential equations is made directly and the solution is relatively simple to obtain, while those in Refs. [9,10] require relatively restrictive criteria. Therefore, the class of mdof nonlinear systems included in Eq. (14) is relatively larger than that previously exists in the literature. Furthermore, in the foregoing derivation the Caughey–Wu form [4] has not been employed.

Remark 2.2. By virtue of Eq. (12), the potential energy of the mdof nonlinear system is given by

$$U = \int \beta_i g_i \, \mathrm{d}x_i, \quad i = 1 \text{ or } 2 \ldots \text{or } n \tag{15}$$

such that Eq. (1) can also be written as

$$\ddot{y}_i + \alpha_{ii}\dot{y}_i + \frac{1}{\beta_i}\frac{\partial U}{\partial y_i} = w_i(t), \tag{16}$$

where now the subscript $i = 1, 2, \ldots, n$.

One can also express

$$H = \frac{1}{2}\left(\sum_{i=1}^{n} x_{n+i}^2\right) + \gamma(x_1, x_2, \ldots, x_n), \tag{17}$$

where the second term on the rhs is related to the potential energy of the system defined by Eq. (15). Eq. (15), when all the factors β_i are equal to unity, is equal to the second term on the rhs of Eq. (17). For illustration, this expression will be explicitly derived in Section 4. Thus, the function H in Eq. (17) is not proportional to ϕ in Eq. (14). In other words, ϕ in Eq. (14) is different from that provided in Refs. [4,5], for example.

Remark 2.3. Generalization to nonlinear systems with parametric stationary random excitations of the white noise type is simple. For example, if the white noise excitations, $w_i(t)$ on the rhs of Eq. (16), are replaced with $\sigma_i(H)w_i(t)$, which contain the parametric random excitation terms through the coefficients $\sigma_i(H)$ of the excitations, the coefficients in Eq. (14) become

$$\beta_i = \frac{\alpha_{ii}}{(\sigma_i^2 D_i)}.$$

3. SNL technique

A two-stage approach is adopted in this section. The first stage is to find the equivalent factors of the damping terms and the second stage is to determine the exact jspdf of the equivalent nonlinear system by applying the results in Section 2. In the second stage, a coordinate transformation is essential. One should note that there are recent publications on equivalent nonlinearization for mdof nonlinear systems under Gaussian white noise excitations [9,11,12]. In particular, Ref. [11] is based on the exact solution of Zhu and Huang [10] mentioned in Section 2. Three criteria for the equivalent nonlinearization technique presented in Ref. [11] are required and therefore it is different from that to be presented in this section. On the other hand, in Ref. [12] the equivalent nonlinearization is hinged also on the exact solutions for systems with linear dampings since the damping coefficients are polynomials of total energy functions of the nonlinear systems. As the total energy of a nonlinear system is assumed to be time invariant the dampings considered in Ref. [12] are linear. In addition, the equivalent nonlinearization technique is based on the minimization of the mean square of the difference between given and approximated dampings. Consequently, the nonlinearization techniques in Refs. [9,11,12] are different from the one to be presented in the following. In addition, they require relatively more algebraic manipulations and therefore the SNL technique to be presented in the following is simpler.

Consider a general mdof nonlinear system governed by the equations of motion

$$\ddot{y}_i + h_i(y_1, y_2, \ldots, y_n, \dot{y}_1, \dot{y}_2, \ldots, \dot{y}_n) = w_i(t), \tag{18}$$

where $i = 1, 2, \ldots, n$ whereas h_i are nonlinear functions of $y_1, y_2, \ldots, y_n$, $\dot{y}_1, \dot{y}_2, \ldots,$ and $\dot{y}_n$.

In the first stage of solution an equivalent nonlinear system to that described by Eq. (18) is required. Let the equations of motion of the equivalent system be

$$\ddot{y}_i + f_i(H)\dot{y}_i + \frac{1}{\beta_i}\frac{\partial U(y_1, y_2, \ldots, y_n)}{\partial y_i} = w_i(t), \tag{19}$$

where $i = 1, 2, \ldots, n$ while β_i has been defined in Section 2; $f_i(H) = \alpha_{ii}(H)$ or simply $f_i = \alpha_{ii}$ are the damping coefficients of the equivalent nonlinear system. Note that f_i will be obtained in the following.

Clearly, there exists a deficiency between every pair of equations of the given system in Eq. (18) and the equivalent nonlinear system in Eq. (19). The equation pairwise deficiency $E_i(Y, \dot{Y})$ is defined as

$$E_i(Y, \dot{Y}) = f_i(H)\dot{y}_i + \frac{1}{\beta_i}\frac{\partial U(Y)}{\partial y_i} - h_i(Y, \dot{Y}), \tag{20}$$

where $Y = [y_1, y_2, \ldots, y_n]$, with $i = 1, 2, \ldots, n$.

$E_i(Y, \dot{Y})$ is transformed into $E_i(H, \theta_1, \theta_2, \ldots, \theta_n)$ and the average of the square of the equation pairwise deficiency over the phases in every cycle is written as

$$I_i = \int_0^{2\pi} \cdots \int_0^{2\pi} [E_i(H, \theta_1, \theta_2, \ldots, \theta_n)]^2 \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n. \tag{21}$$

Then, I_i is minimized with respect to f_i. That is,

$$\frac{\mathrm{d}I_i}{\mathrm{d}f_i} = 0,$$

which results in the factors associated with the damping terms as

$$f_i(H) = \frac{\int_0^{2\pi} \cdots \int_0^{2\pi} [(\dot{y}_i)h_i - (\dot{y}_i)g_i]\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n}{\int_0^{2\pi} \cdots \int_0^{2\pi} (\dot{y}_i)^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n}, \tag{22}$$

where $i = 1, 2, \ldots, n$ while $h_i = h_i(H, \theta_1, \theta_2, \ldots, \theta_n)$ and $g_i = (1/\beta_i)\partial U/\partial y_i = g_i(H, \theta_1, \theta_2, \ldots, \theta_n)$, in addition, y_i and $\dot{y}_i$ are functions of $H, \theta_1, \theta_2, \ldots, \theta_n$. Note that Eq. (22) is the essence of the present SNL technique which is entirely different from those in Refs. [9,11,12].

Before proceeding further it should be mentioned that $\mathrm{d}I_i/\mathrm{d}f_i = 0$ is a minimum as

$$\frac{\mathrm{d}^2 I_i}{\mathrm{d}f_i^2} = \int_0^{2\pi} \cdots \int_0^{2\pi} (\dot{y}_i)^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n \tag{23}$$

is always positive as long as H is real.

The second stage of solution of the present SNL technique is to obtain the exact joint stationary probability density function (jspdf) of the equivalent system governed by Eq. (19). It is defined by Eq. (14).

Before ending this section, it should be mentioned that generalization of the SNL technique to include parametric stationary white noise excitations along the line presented in Remark 2.3 is simple but will not be considered presently for brevity.

4. Application and comparison

In order to demonstrate the simplicity of the proposed SNL technique and its accuracy, the tdof nonlinear system governed by the following equations of motion is considered:

$$\begin{aligned} &\ddot{y}_1 - (\lambda_1 - \zeta_1 \dot{y}_1^2)\dot{y}_1 + \omega_1^2 y_1 + a y_2 + b(y_1 - y_2)^3 = w_1(t),\\ &\ddot{y}_2 - (\lambda_1 - \lambda_2 - \zeta_2 \dot{y}_2^2)\dot{y}_2 + \omega_2^2 y_2 + a y_1 + b(y_2 - y_1)^3 = w_2(t), \end{aligned} \tag{24}$$

where a, b, ζ_i, λ_i, and ω_i with $i = 1, 2$, are constant, and the remaining symbols have already been defined above. Note that the above nonlinear system contains nonlinear damping terms and nonlinear stiffness terms simultaneously. Thus, no exact solution is available. The approximate solution for this system has been obtained by applying another SNL technique [2] in which the equivalent damping coefficient is different from the present one defined by Eq. (22). Furthermore, in Ref. [2] no comparison was made to results by using MCS.

In the first stage of solution in the presently proposed SNL technique, the following equivalent system of equations is required:

$$\ddot{y}_1 + f_1(H)\dot{y}_1 + g_1 = w_1(t), \quad \ddot{y}_2 + f_2(H)\dot{y}_2 + g_2 = w_2(t), \tag{25}$$

where g_i with $i = 1, 2$ are given as

$$g_1 = \omega_1^2 y_1 + a y_2 + b(y_1 - y_2)^3, \quad g_2 = \omega_2^2 y_2 + a y_1 + b(y_2 - y_1)^3, \tag{26a,b}$$

such that the potential energy of the system can be shown as

$$U(y_1, y_2) = \tfrac{1}{2}\beta_2 \omega_1^2 y_1^2 + \tfrac{1}{2}\beta_2 \omega_2^2 y_2^2 + a\beta_2 y_1 y_2 + \tfrac{1}{4} b\beta_2 (y_1 - y_2)^4, \tag{26c}$$

in which the parameter β_i is defined in Eq. (7).

The coordinate transformation selected for this tdof nonlinear system is

$$\dot{y}_1 = \sqrt{2H}\,\sin\theta_1 \cos\theta_2, \quad \dot{y}_2 = \sqrt{2H}\,\sin\theta_1 \sin\theta_2 \tag{27a,b}$$

and

$$\gamma(y_1, y_2) = \frac{1}{2}\sum_{i=1}^{2} \omega_i^2 y_i^2 + a y_1 y_2 + \frac{b}{4}(y_2 - y_1)^4 \tag{27c}$$

or its transformed version

$$\gamma(H, \theta_1, \theta_2) = \frac{b}{4}(R_1 + R_2), \tag{27d}$$

in which

$$R_1 = \left(\frac{4H}{b}\cos^2\theta_1 \cos^2\theta_2\right), \quad R_2 = \left(\frac{4H}{b}\cos^2\theta_1 \sin^2\theta_2\right). \tag{27e,f}$$

By applying the above coordinate transformation to Eq. (17), one can show that Eq. (17) is satisfied.

By virtue of Eq. (22) one can show that

$$f_1(H) = \tfrac{9}{8}(\zeta_1 H) - \lambda_1. \tag{28a}$$

Similarly, one can obtain

$$f_2(H) = \tfrac{9}{8}(\zeta_2 H) - \lambda_1 + \lambda_2. \tag{28b}$$

Having obtained Eq. (28) one can evaluate β_i, in which $\alpha_{ii} = f_i(H)$. Consequently, the exact jspdf of the equivalent system can be determined by Eq. (14). With the system parameters $a = 1$, $b = \varepsilon = 0.1,\ 0.3$, $\zeta_1 = 0.1$, $\zeta_2 = 0.2$, $\lambda_1 = 1.0$, $\lambda_2 = 3.0$, $S_1 = 1.0/(2\pi)$, $S_2 = 1.0/(2\pi)$, and $\omega_i = 1.0$, with $i = 1, 2$, one obtains the exact jspdf of the equivalent tdof nonlinear system as Eq. (14), where now

$$\phi = \frac{1}{2}\left(\sum_{i=1}^{2} \beta_i x_{n+i}^2\right) + U(x_1, x_2)$$

or transforming back to the original coordinate system,

$$\phi = \frac{1}{2}(\beta_1 \dot{y}_1^2 + \beta_2 \dot{y}_2^2) + U(y_1, y_2), \tag{29}$$

in which $U(y_1, y_2)$ is defined by Eq. (26c) and $\beta_1 \neq \beta_2$.

For brevity, representative computed results of the exact jspdf of the equivalent nonlinear system obtained by the present SNL technique are compared with those of MCS and included in

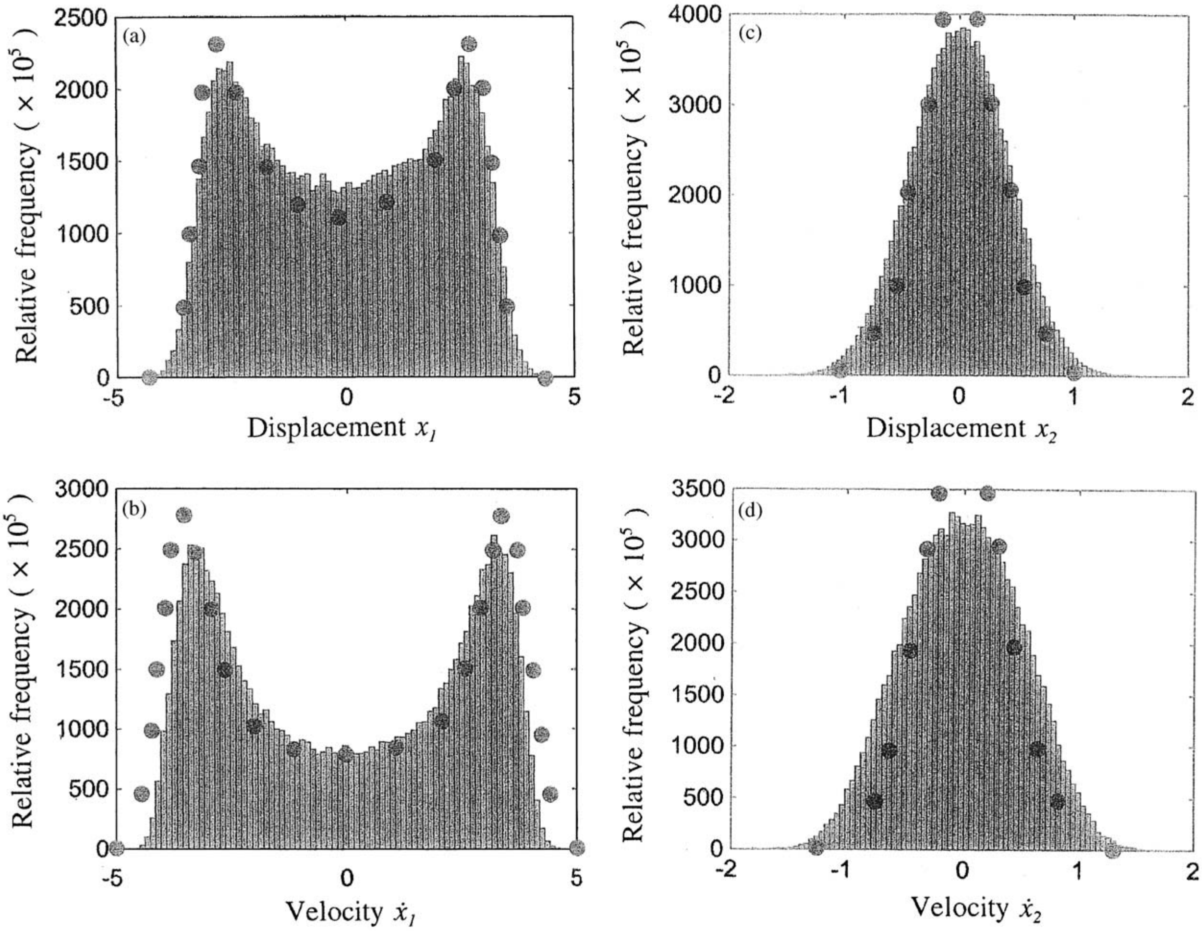

Fig. 1. Frequency histogram and joint probability density of tdof nonlinear system with $\zeta_1 = 0.1$, $\zeta_2 = 0.2$, $\varepsilon = 0.1$, $\lambda_1 = 1.0$, $\lambda_2 = 3$. MCS results (bars), data from present SNL technique (solid circles). (a) Displacement x_1, (b) velocity $\dot{x}_1$, (c) displacement x_2, and (d) velocity $\dot{x}_2$.

Figs. 1 and 2. The MCS results are obtained by using the computer package MATLAB version 6.5. It is interesting to note that owing to the nonlinear damping terms in the system, the responses are non-Gaussian even when the applied stationary white noise excitations are Gaussian. During computational experiments, it was observed that for the above system parameters with $b = 0$, the MCS solution for displacements were unstable. However, when the Gaussian white noise excitation $w_1(t)$ was changed to $(2)^{1/2}w_1(t)$, the displacements from the MCS were stable. This indicates that when $b = 0$ the system displacement responses are very sensitive to the change of magnitudes of Gaussian white noise excitations. The proposed SNL technique can capture the non-Gaussian character of all responses. With reference to all the cases considered in the computational experiments and those presented in Figs. 1 and 2, one can conclude that the present SNL technique for mdof nonlinear systems can give very accurate results compared with those from MCS.

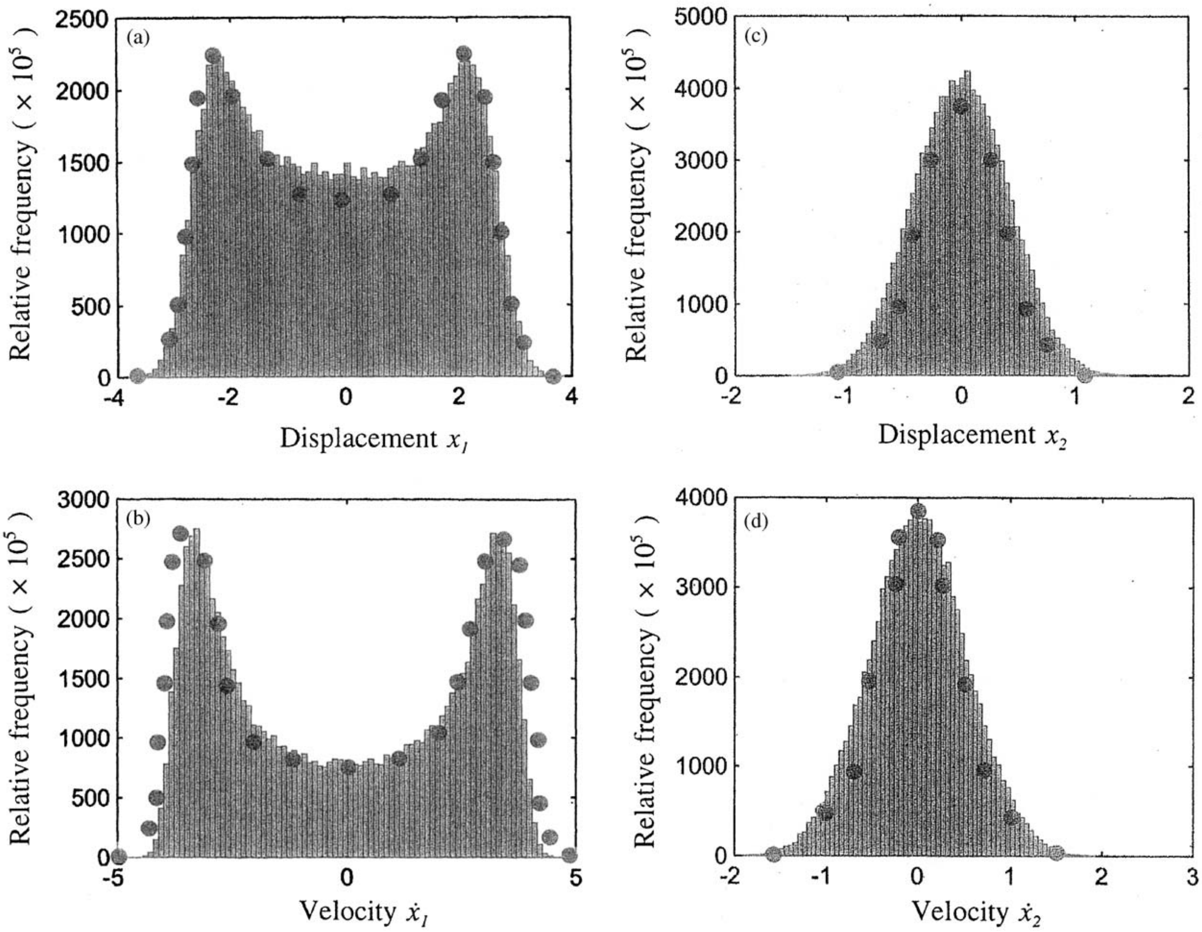

Fig. 2. Frequency histogram and joint probability density of tdof nonlinear system with $\zeta_1 = 0.1$, $\zeta_2 = 0.2$, $\varepsilon = 0.3$, $\lambda_1 = 1.0$, $\lambda_2 = 3$. MCS results (bars), data from present SNL technique (solid circles). (a) Displacement x_1, (b) velocity $\dot{x}_1$, (c) displacement x_2, and (d) velocity $\dot{x}_2$.

It may be also appropriate to note that while MCS results and approximate solutions for marginal stationary probability densities were included as three-dimensional plots in Ref. [11] they were not superimposed on one another and therefore direct comparison between the MCS and approximate solutions cannot be made. A closer inspection on the magnitudes of the marginal stationary probability densities in Figs. 2–5 of Ref. [11] reveals that the MCS results and approximate solutions are significantly different.

5. Concluding remarks

A statistical nonlinearization (SNL) technique has been presented for the solution of the joint probability density function of general multi-degrees of freedom (mdof) nonlinear systems under stationary white noise excitations. It consists of two stages in the solution. The first stage of solution is the determination of the equivalent damping factors and the second stage is the application of the exact joint stationary probability density function (jspdf) for the equivalent nonlinear system. The exact solution is different from that of Caughey [4,5], for example, in that the ratios of damping coefficients to intensities of applied white noise excitations are not identical and therefore it is new. The present exact solution is also different from those of Cai and Lin [9], and Zhu and Huang [10] in that it is obtained directly from the theory of differential equations while that of Cai and Lin [9] requires satisfaction of a relatively restrictive criterion, and that of Zhu and Huang [10] adopts the Hamiltonian formulation such that their exact solution depends on the number of independent integrals of motion, for example. The proposed SNL technique is simple to implement compared with those in Refs. [9,11,12] since the equivalent damping coefficients defined by Eq. (22) can be easily derived.

Computed results by applying the proposed SNL technique for a general two-degrees of freedom (tdof) nonlinear system are compared with those obtained by the Monte Carlo simulation (MCS). With reference to the obtained results and those presented in Figs. 1 and 2, one can conclude that the SNL technique is simple to implement, very accurate, and is applicable to mdof systems with both nonlinear damping and nonlinear restoring forces. It is applicable to systems with large nonlinearities and large intensities of excitations.

References

[1] C.W.S. To, A statistical non-linearization technique in structural dynamics, *Journal of Sound and Vibration* 161 (3) (1993) 543–548.

[2] C.W.S. To, *Nonlinear Random Vibration: Analytical Techniques and Applications*, Swets & Zeitlinger Publishers, Lisse, The Netherlands, 2000.

[3] C.W.S. To, Exact and equivalent solutions of two-degree-of-freedom nonlinear systems under stationary white noise excitations, in: W.Q. Zhu, G.Q. Cai, R.C. Zhang (Eds.), *Advances in Stochastic Structural Dynamics*, Proceedings of the Fifth International Conference on Stochastic Structural Dynamics, Hangzhou, China, August 11–13, 2003, pp. 445–452.

[4] T.K. Caughey, Derivation and application of the Fokker–Planck equation to discrete nonlinear dynamic systems subjected to white random excitation, *Journal of the Acoustical Society of America* 35 (11) (1963) 1683–1692.

[5] T.K. Caughey, Nonlinear theory of random vibration, *Advances in Applied Mechanics* 11 (1971) 209–253.

[6] Y.K. Lin, G.Q. Cai, *Probabilistic Structural Dynamics: Advanced Theory and Applications*, McGraw-Hill, Inc., New York, 1995.

[7] C. Soize, Exact stationary response of multi-dimensional non-linear Hamiltonian dynamical systems under parametric and external stochastic excitations, *Journal of Sound and Vibration* 149 (1) (1991) 1–24.

[8] W.Q. Zhu, Y.K. Lin, On exact stationary solutions of stochastically perturbed Hamiltonian systems, *Probabilistic Engineering Mechanics* 5 (2) (1990) 84–87.

[9] G.Q. Cai, Y.K. Lin, Exact and approximate solutions for randomly excited mdof non-linear systems, *International Journal of Non-Linear Mechanics* 31 (5) (1996) 647–655.

[10] W.Q. Zhu, Z.L. Huang, Exact stationary solutions of stochastically excited and dissipated partially integrable Hamiltonian systems, *International Journal of Non-Linear Mechanics* 36 (1) (2001) 39–48.

[11] W.Q. Zhu, Z.L. Huang, Y. Suzuki, Equivalent non-linear system method for stochastically excited and dissipated partially integrable Hamiltonian systems, *International Journal of Non-Linear Mechanics* 36 (2001) 773–786.

[12] L. Cavaleri, M. Di Paola, G. Failla, Some properties of multi-degree-of-freedom potential systems and application to statistical equivalent non-linearization, *International Journal of Non-Linear Mechanics* 38 (2003) 405–421.